Calculus – Chapter 2 Review – The Theorems and Definitions

Definition of limit: We say <u>the limit of $f(x)$ as x approaches a exists and equals L</u>, or, in more compact notation, $\lim_{x\to a} f(x) = L$, if, given any small interval centered on L, say $(L-\varepsilon, L+\varepsilon)$ we can produce a small interval centered on a, say $(a-\delta, a+\delta)$ such that if $a-\delta \le x \le a+\delta$ then $L-\varepsilon \le f(x) \le L+\varepsilon$. That is, by taking x close enough to a, we can assure that $f(x)$ is arbitrarily close to L.

Note: As far as $\lim_{x\to a} f(x)$ is concerned, it doesn't matter at all what the value of the $f(a)$ is, or even whether or not it is defined. All that matters is the behavior of $f(x)$ in the neighborhood of (a,L) .

One-Sided Limits: We denote the limiting value of $f(x)$ as x approaches a through values greater than a as the right-handed limit, $\lim_{x\to a^+} f(x) = L_1$ while the limiting value of $f(x)$ as x approaches a through values less than a is the left-handed limit, $\lim_{x\to a^-} f(x) = L_2$. $\lim_{x\to a} f(x)$ exists if and only if both the left and right limits exist and agree; that is, if and only if $L_1 = L_2$.

The properties of the limits follow mostly from the properties of real numbers.

Definition of Continuity: Oftentimes, the simplest way to evaluate a limit $\lim_{x\to a} f(x)$ is to just evaluate $f(a)$, this is true for functions continuous at the point $(a, f(a))$. In short, f is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

Note: Continuity is a stronger condition than the existence of the limit.

Definition of Continuous Function: Many well-known functions are continuous at every point in their domain. These are called *continuous functions*. The standard polynomial, rational, trigonometric and exponential functions are continuous, as are their inverses. That is, if a is in the domain of a continuous function f, then $\lim_{x\to a} f(x) = f(a)$.

Claim: If f is a one-to-one function continuous at $(a, f(a))$, then f^{-1} is continuous at $(f(a), a)$. Can you prove that?

1. Consider $f(x) = \dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$
    a. Approximate the value of $f(x)$ at x = 64.01, 64.001, 63.99 and 63.999 – what do your results suggest about $\lim_{x\to 64} f(x)$?
    b. How close does x have to be to 64 to ensure that the function is within 0.1 of it's limit?

A *lemma* is a relatively simple theorem that is used to prove a more complicated theorem. One example might be the following:

Lemma: If $f(x) \le g(x)$ in an open neighborhood of $x = a$ and both limits exist then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. This can be proved as a consequence of the definition of the limit. Try proof by contradiction.

The Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

The proof of this theorem follows from the lemma above. That's the idea of a lemma.

Under what conditions can we be sure that $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$?

Theorem: It is sufficient that $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = f(b)$ to conclude the above.

Intuitive justification: By definition of the limit, if *x* is close enough to *a*, then g(*x*) is arbitrarily close to *b* , and if the neighborhood is truly small enough then since $\lim_{x\to b} f(x) = f(b)$, it follows that $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$.

2. Is there a number *a* such that $\lim_{x\to 1} \frac{2x^2 + ax + a}{x^2 + x - 2}$ exists? If not, why not? If so, find the value of *a* and the value of the limit.

3. Consider

Theorem: In the above theorem, strengthening the condition $\lim_{x\to a} g(x) = b$ to the stronger condition that *g* is continuous at *x*=*a*: $\lim_{x\to a} g(x) = g(a)$, allows us to make the same conclusion

Can you think of an example that shows how the condition $\lim_{x\to a} g(x) = g(a)$ is not necessary for the conclusion but $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$?

The Intermediate Value Theorem: If *f* is continuous at every point on the closed interval [*a*,*b*] then for all *N* between *f*(*a*) and *f*(*b*) there is some *c* in (*a*,*b*) such that *f*(*c*) = *N*.

The proof is surprisingly difficult. You can find various proofs on the internet, such as at
http://planetmath.org/encyclopedia/ProofOfIntermediateValueTheorem.html
http://www.brainyencyclopedia.com/encyclopedia/i/in/intermediate_value_theorem.html
http://www.cs.uleth.ca/~holzmann/notes/intermediate.pdf

All limits involve either infinity or the infinitesimal. Here's a definition of infinitesimal from Merriam Webster: Main Entry: ^{2}**infinitesimal**
Function: *adjective*
1 : taking on values arbitrarily close to but greater than zero
2 : immeasurably or incalculably small
- **in·fin·i·tes·i·mal·ly** /-m&-lE/ *adverb*

Limits Involving Infinity

1. $\lim_{x\to a} f(x) = \infty$ if for any N, no matter how large, we can find δ>0 such that if $a-\delta \le x \le a+\delta$ then $f(x) > N$.
2. If any of the following are true, then we say that f has a vertical asymptote at $x = a$:
$\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to a^-} f(x) = -\infty$
3. If $\lim_{x\to \infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$ then f has a horizontal asymptote along $y = L$.

Secant and Tangent lines

The line through $(a, f(a))$ and $(a+h, f(a+h))$ is called a secant line for f and we can compute its slope as usual: $m_{\sec} = \dfrac{f(a+h)-f(a)}{h}$. If $\lim_{h\to 0} m_{\sec}$ exists then we say f has a tangent line at $(a, f(a))$ and its equation is $y = f(a) + f'(a)(x-a)$ where $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

The Derivative Function:

The function defined by $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, wherever this limit exists, is called the derivative function of f and gives the slope of the line tangent to f at $(x, f(x))$, provided there is a tangent line. It can also be thought of as the instantaneous rate of change of input per output.

Note: This limit is equivalent to $f'(x) = \lim_{a\to x} \dfrac{f(x)-f(a)}{x-a}$ or $f'(x) = \lim_{a\to x} \dfrac{f(x+h)-f(x-h)}{2h}$, each of which describe the limiting value of slopes of secant lines as the two points describing the secant line converge on $(x, f(x))$. See figures on page 151 of Stewart, *Concepts and Context*.

Alternate notations for the derivative function include $\dfrac{d}{dx} f(x) = \dfrac{d}{dx} y = \dfrac{dy}{dx} = D_x y = D_x f(x)$.

Definition: A function f is <u>differentiable</u> at a iff $f'(a) = \lim_{\mho\to a} \dfrac{f(a)-f(\mho)}{a-\mho}$ exists. The function f is differentiable on the interval (a,b) iff it's differentiable at every point on the interior of the interval (a,b).

Differentiability is a stronger condition than continuity (which is a stronger condition than the existence of the limit). That is, if $f'(a)$ exists then *f* is continuous at $x = a$. Try to prove this on your own without reviewing the proof given in the text, p 163.

Linear approximations: If *f* is differentiable at $x = a$ then $y = f(a) + f'(a)(x - a)$ is a good approximation to *f* near $(a, f(a))$. For example, since the slope of the line tangent to $f(x) = \sqrt{x}$ at x =100 is 1/20, $\sqrt{105} \approx \sqrt{100} + \frac{1}{20}(105 - 100) = 10.25$

What Does the Derivative Function Say About the Function?

- Since the derivative of a constant is zero, any vertical shift of $y = f(x)$ will have the same derivative function.
- If $f'(a) = 0$ then the tangent line at $(a, f(a))$ is horizontal.
- If $f'(x) > 0$ on an interval, then *f* is increasing on that interval.
- If $f'(x) < 0$ on an interval, then *f* is decreasing on that interval.
- If $f''(x) > 0$ on an interval, then *f* is concave up on that interval.
- If $f''(x) < 0$ on an interval, then *f* is concave down on that interval.
- If $\lim_{x \to \infty} f'(x) = 0$ then $y = f(x)$ has a horizontal tangent line.
- If $\lim_{x \to \infty} f'(x) = a \neq 0$ then $y = f(x)$ has a slant asymptote.